Journal of Statistical Physics, Vol. 96, Nos. 1/2, 1999

Variational Principle for Some Renormalized Measures

R. Lefevere[1]

Received October 8, 1998; final March 8, 1999

We show that some measures suffering from the so-called renormalization group pathologies satisfy a variational principle and that the corresponding limit of the pressure, with boundary conditions in a set of measure 1, is equal to the pressure of the Ising model modulo a scale factor.

KEY WORDS: Renormalization group pathologies; variational principle; weakly Gibbsian; Gibbs measures.

1. INTRODUCTION

The general problem of determining whether a given measure is an equilibrium distribution for an infinite collection of microscopic components in interaction has recently received much attention, in particular in real-space Renormalization Group theory.[7–9, 20, 21, 19, 22, 18] In the work of van Enter, Fernandez and Sokal,[21] many examples of so-called Renormalization Group pathologies are exhibited, which suggest that a Gibbsian description (i.e. in terms of microscopic interactions) of many measures obtained by a Renormalization Group transformation (RGT) from a low-temperature Ising measure may prove to be impossible.

Subsequently, there have been many successful attempts to bring several examples considered in ref. 21 back into the Gibbslan framework. One direction has been to show that after sufficiently many iterations of the transformations the measure obtained is again Gibbsian in the standard sense.[15, 16] Another direction, suggested by Dobrushin,[2] has been to relax the constraints imposed on the interactions, roughly speaking the interaction of one microscopic component with all the others is no more required to be finite for *every* microscopic configurations but only for *almost every*

[1] UCL, Physique Théorique, B-1348, Louvain-la-Neuve, Belgium; e-mail: lefevere@fyma.ucl.ac.be.

0022-4715/99/0700-0109$16.00/0

(with respect to the measure under study) configurations. It has been shown[14] that the projected measures of the Schonmann[17] example can be described with the help of such a potential and in ref. 1, it is shown that several RGT examples of ref. 21 are also Gibbsian with a potential satisfying this weaker constraint, the pathologies being in some sense analogous to the Griffith singularities.[6] As this potential is not in the "Israel's big Banach space," the standard thermodynamic theory does not apply and our goal here is to investigate its thermodynamic properties, as suggested in ref. 12. In particular, we prove the existence of the pressure (with boundary conditions in a large set) and of the average energy density for a class of tempered measures. We also prove a variational principle for the renormalized measures considered in ref. 1 and thereby obtain a nice characterization of those measures. The situation here is somewhat similar to the case of unbounded spins systems,[11, 10] if one makes an analogy between the size of a "bad cluster" in our case and the value of an unbounded spin. In the case of unbounded spins, one imposes a condition of moderate growth of the boundary condition, in order to prove the existence of the thermodynamic limit of the free energy; we obtain a completely similar condition on the growth of the bad clusters. Moreover, in order to formulate a variational principle, as the potential is not uniformly absolutely summable, it is necessary to introduce a class of tempered measures such that the expectation value of the energy density is finite. These tempered measures are characterized by the fact that they give a small probability to the occurence of a large "bad cluster" around any given point.

The paper is organized as follows; in Section 2, we give the definition of the notion of "weakly" Gibbsian, briefly recall the results obtained in ref. 1 and define the appropriate thermodynamic quantitites. In Section 3, we outline the construction of the potential of ref. 1 and introduce a class of tempered measures. In Section 4, we state our main results. Section 5 is devoted to the proof of our main result and in the Appendix we collected the proofs of the lemmas and the proof of some statements in the main text that can be easily derived from ref. 1.

2. DEFINITIONS

We follow closely the notations and notions introduced in ref. 1. We consider the nearest-neighbour Ising model on $\mathbf{Z}^d$, $d \geqslant 2$, at β large, for simplicity. To each $i \in \mathbf{Z}^d$, we associate a variable $\sigma_i \in \{-1, +1\}$, and the (formal) Ising Hamiltonian is

$$-\beta H^I = \beta \sum_{\langle ij \rangle} (\sigma_i \sigma_j - 1) \tag{1}$$

where $\langle ij \rangle$ denotes a nearest-neighbour pair and β is the inverse temperature. At low temperatures, there are two extremal translation invariant Gibbs measures corresponding to (1), μ_+ and μ_-. To define our RGT, let $\mathscr{L} = (b\mathbf{Z})^d$, $b \in \mathbf{N}$, $b \geqslant 2$ and cover $\mathbf{Z}^d$ with disjoint *b-boxes* $B_x = B_0 + x$, $x \in \mathscr{L}$ where B_0 is a box of size b centered around 0. Associate to each $x \in \mathscr{L}$ a variable $s_x \in \{-1, +1\}$, denote by σ_A an element of $\{-1, +1\}^A$, for $A \subset \mathbf{Z}^d$, $|A| < \infty$, and introduce the probability kernels

$$T_x = T(\sigma_{B_x}, s_x)$$

for $x \in \mathscr{L}$, i.e. T_x satisfies

$$\begin{aligned} &(1) \quad T(\sigma_{B_x}, s_x) \geqslant 0 \\ &(2) \quad \sum_{s_x} T(\sigma_{B_x}, s_x) = 1 \end{aligned} \tag{2}$$

We shall use the notation $T(\sigma_{B_V}, s_V) \equiv \prod_{x \in V} T(\sigma_{B_x}, s_x)$.

For any probability measure μ on $\{-1, +1\}^{\mathbf{Z}^d}$ (resp. $\{-1, +1\}^{\mathscr{L}}$), we denote by $\mu(\sigma_A)$ (resp. $\mu(s_A)$) the probability of the configuration σ_A (resp. s_A).

Definition. Given a measure μ on $\{-1, +1\}^{\mathbf{Z}^d}$, the *renormalized measure* μ' on $\Omega = \{-1, +1\}^{\mathscr{L}}$ is defined by:

$$\mu'(s_A) = \sum_{\sigma_{B_A}} \mu(\sigma_{B_A}) \prod_{s \in A} T(\sigma_{B_x}, s_x) \tag{3}$$

where $B_A = \bigcup_{x \in A} B_x$, $A \subset \mathscr{L}$, $|A| < \infty$, and $s_A \in \Omega_A = \{-1, +1\}^A$.

It is easy to check, using (1) and (2), that μ' is a measure. We shall call the spins σ_i the *internal* spins and the spins s_x the *external* ones (they are also sometimes called the block spins).

Two other conditions are imposed on T: we assume that T is symmetric:

$$T(\sigma_{B_x}, s_x) = T(-\sigma_{B_x}, -s_x) \tag{4}$$

and that

$$0 \leqslant T(\sigma_{B_x}, s_x) \leqslant e^{-\beta} \tag{5}$$

if $\sigma_i \neq s_x \ \forall i \in B_x$.

The condition (5) means that there is a coupling which tends to align s_x and the spins in the block B_x. We also define

$$\bar{T} \equiv T(\{\sigma_i = +1\}_{i \in B_x}, +1) \tag{6}$$

The usual transformations, discussed in [21, Sect. 3.1.2], like decimation or majority rule, obviously satisfy (4, 5). The Kadanoff transformation satisfies (5) for p large. As in ref. 1, our results could be extended to the block spin transformations, where s_x does not belong to $\{-1, +1\}$.

Consider potentials (interactions) $\Phi = (\Phi_X)$ which are families of functions

$$\Phi_X \colon \Omega_X \to \mathbf{R} \tag{7}$$

indexed by $X \subset \mathscr{L}$, $|X| < \infty$.

We say that a potential is $\bar{\Omega}$-pointwise absolutely summable, with $\bar{\Omega} \subset \Omega$, if

$$\sum_{X \ni x} |\Phi_x(s_x)| < \infty \qquad \forall x \in \mathscr{L}, \quad \forall s \in \bar{\Omega} \tag{8}$$

Then, if $\bar{\Omega}$ is in the tail-field, one may define a Hamiltonian in a finite volume V with boundary conditions $\bar{s} \in \bar{\Omega}$ by the usual formula:

$$H(s_V \mid \bar{s}_{V^c}) = -\sum_{X \cap V \neq \varnothing} \Phi_X(s_{X \cap V} \vee \bar{s}_{X \cap V^c}) \tag{9}$$

where $s_V \in \Omega_V$, $\bar{s}_{V^c}$ is the restriction to V^c of $\bar{s} \in \bar{\Omega}$ and, for $X \cap Y = \varnothing$, $s_X \vee s_Y$ denotes the obvious configuration in $\Omega_{X \cup Y}$ we need the fact that $\bar{\Omega}$ is in the tail-field because then if $\bar{s} \in \bar{\Omega}$ then for all finite V and all s, one has $s_V \vee \bar{s}_{V^c} \in \bar{\Omega}$. In particular, we denote by $H(s_V \mid +)$, the Hamiltonian with the configuration $\bar{s}$ such that $s_x = +1$ for all $x \in \mathscr{L}$ and $H(s_V \mid f) = -\sum_{X \subset V} \Phi_X(s_X)$, the Hamiltonian with free boundary conditions.

The main result of ref. 1 is the following, for β large enough, the renormalized measures μ'_+ and μ'_- are "Gibbsian" in the following sense: there exist a translation-invariant set $\bar{\Omega}$ (hence in the tail-field), and a translation invariant $\bar{\Omega}$-pointwise absolutely summable interaction Φ such that $\mu'_+(\bar{\Omega}) = \mu'_-(\bar{\Omega}) = 1$ and for μ'_+ and μ'_- there exists a version of the conditional probabilities that satisfy $\forall V \subset \mathscr{L}$, $|V|$ finite, $\forall s_V \in \Omega_V$

$$\mu'(s_V \mid \bar{s}_{V^c}) = \begin{cases} Z^{-1}(\bar{s}_{V^c}) \exp(-H(s_V \mid \bar{s}_{V^c})) & \text{for} \quad \bar{s} \in \bar{\Omega} \\ 0 & \text{for} \quad \bar{s} \notin \bar{\Omega} \end{cases} \tag{10}$$

In the terminology introduced in ref. 12, the measures μ'_+ and μ'_- are said to be *weakly* Gibbsian for the potential Φ.

Let us now define the thermodynamic quantitites that we shall consider. The partition function corresponding to the above Hamiltonian with boundary conditions $\bar{s} \in \Omega$ is obviously given by:

$$Z_V^{\bar{s}} = Z_V(\bar{s}_{V^c}) = \sum_{s_V} \exp -H(s_V \mid \bar{s}_{V^c}) \tag{11}$$

and we shall investigate the existence of the thermodynamic limit,

$$p^{\bar{s}} = \lim_{V \uparrow \mathbf{Z}^d} \frac{1}{|V|} \log Z_V^{\bar{s}} \tag{12}$$

defining the pressure with fixed boundary conditions. The pressure of the Ising model on the original lattice is denoted by p_{Ising} we also need to define the relative entropy of two probability measures and give its basic properties. Given two probability measures μ_1, μ_2 on $\{-1, +1\}^{\mathbf{Z}^d}$, we define the finite volume relative entropy for a volume V in $\mathbf{Z}^d$ as follows:

$$S_V(\mu_1 \mid \mu_2) = \sum_{s_V} \mu_1(s_V) \log \frac{\mu_1(s_V)}{\mu_2(s_V)} \tag{13}$$

One can show that $S_V(\mu_1 \mid \mu_2) \geqslant 0\ \forall \mu_1, \mu_2$ and that $S_V(\mu_1 \mid \mu_2) = 0$ iff the restriction of μ_1 and μ_2 to V coincide. Moreover, it can be proven (see Lemma 3.3 in ref. 21) that for any stochastic transformations T defined by (2) (including the deterministic case where $T(\sigma_{B_x}, s_x) = 1$ for some σ_{B_x}) and any measures μ_1, μ_2, we have:

$$S_V(\mu_1 \mid \mu_2) \geqslant S_V(\mu'_1 \mid \mu'_2) \tag{14}$$

We shall need also the following property: consider μ_1 and μ_2, two finite volume Gibbs measures for the Ising Harniltonian with stochastic boundary conditions, i.e. μ_i is given by

$$\mu_i(\sigma_V) = \sum_{\bar{\sigma}_{Vc}} \frac{\exp -\beta H^I(\sigma_V \mid \bar{\sigma}_{V^c})}{Z(\bar{\sigma}_{V^c})} \nu_i(\bar{\sigma}_{V^c}) \tag{15}$$

with ν_i a probability measure on $\{-1, +1\}^{\mathbf{Z}^d}$. Then,

$$\lim_{V \uparrow \mathbf{Z}^d} \frac{1}{|V|} \sum_{\sigma_V} \mu_1(\sigma_V) \log \frac{\mu_1(\sigma_V)}{\mu_2(\sigma_V)} = 0 \tag{16}$$

The proof of this property is easy: divide the numerator and the denominator of the ratio in (16) by the finite volume Gibbs measure with free boundary conditions; then, because one has

$$|H^I(\sigma_V \mid \bar{\sigma}_{V^c}) - H^I(\sigma_V \mid f)| \leqslant 2\,|\partial V| \tag{17}$$

for all σ_v and $\bar{\sigma}_{V^c}$ (where ∂V denotes the set of sites $\{x \in V \mid d(x, V^c) = 1\}$), the result follows.

The entropy density of a measure μ is given by the following formula:

$$s(\mu) = \lim_{V \uparrow \mathbf{Z}^d} \frac{1}{|V|} S_V(\mu) \equiv \lim_{V \uparrow \mathbf{Z}^d} \frac{1}{|V|} \sum_{s_V} \mu(s_V) \log \mu(s_V) \tag{18}$$

this limit is well defined and finite for any translation invariant μ on $\{-1, +1\}^{\mathbf{Z}^d}$.[5]

3. CONSTRUCTION OF THE POTENTIALS AND THE TEMPERED MEASURES

Here and in the following sections, we shall denote by c or C a generic constant that depends only on the dimension or on b, but not on the choice of the parameters L and α (see below). $C(L)$ or $C(\varepsilon)$ will denote some generic constants depending on their arguments. The value of all of those constants may change from place to place.

3.1. The Potentials

Let us now recall briefly the construction of the potentials and of the set $\bar{\Omega}$ obtained in ref. 1, we refer to that paper for further details. First, one introduces a (random) subset of $\mathscr{L}$

$$D(s) = \bigcup \{B_x \mid x \in \mathscr{L}, \exists y \in \ell, |x-y| = b, s_x \neq s_y\} \tag{19}$$

which is described on different coarse grained scales, as we shall see below. We also define $D_V(s_V)$ by replacing $\mathscr{L}$ by V in the above definition. Let $\alpha > 0$ and $L > b$ be some odd integer (which will be taken large enough so that inequalities like $C < L^\alpha$ hold). Divide $\mathbf{Z}^d$ into disjoint L-boxes $\{i \mid |i - Lx| < L/2\}$ where $x \in \mathbf{Z}^d$, i.e. each $i \in \mathbf{Z}^d$ can be written as $i = Lx + j$ with $x \in \mathbf{Z}^d$ and $|j_\mu| < L/2$, $\mu = 1,..., d$ (here and below, we use the letters x, y to denote sites in the new lattices $\mathbf{Z}^d$). we define $[L^{-1}i] = x$ and the L-box of sites i such that $[L^{-1}i] = x$ is denoted by Lx. Also for a set $Y \subset \mathbf{Z}^d$, $LY = \bigcup \{Lx \mid x \in Y\}$. We use a similar notation for all scales L^n, $n = 1, 2,...$.

We shall now describe $D(s)$ on these different coarse-grained scales. Let us introduce the random variables $N_x^n = N_x^n(s)$, $n = 0, 1, 2....$, defined inductively as follows:

$$\begin{aligned} N_x^0 &= 2d \qquad \text{if} \quad x \in D(s) \\ N_x^0 &= 0 \qquad \text{otherwise} \end{aligned} \tag{20}$$

$$N_{x'}^{n+1} = L^{-1+4\alpha} \sum_{y \in Lx' \setminus \mathscr{D}^n(s)} N_y^n \tag{21}$$

where

$$\begin{aligned} \mathscr{D}^n(s) = \{ D_i \mid D_i &\text{ is a connected component of } D^n(s), \\ &|D_i| \leqslant L^\alpha, N^n(D_i) \leqslant L^{-3\alpha} \} \end{aligned} \tag{22}$$

with $D^0(s) = D(s)$;

$$N^n(Y) = \sum_{x \in Y} N_x^n \tag{23}$$

for $Y \subset \mathbf{Z}^d$, and

$$D^{n+1}(s) = \overline{[L^{-1}(D^n(s) \setminus \mathscr{D}^n(s))]} \tag{24}$$

where, for a set $Y \subset \mathbf{Z}^d$, we write:

$$\bar{Y} = \{ i \mid d(i, Y) \leqslant 1 \} \tag{25}$$

It is easy to see inductively that

$$N_x^n = 0 \qquad \text{if} \quad x \notin D^n(s) \tag{26}$$

and

$$D^n(s) \subset \overline{\{ x \mid N_x^n \neq 0 \}} \tag{27}$$

We also define variable $N_{x,V}^n$ and sets D_V^n and $\mathscr{D}_V^n$ by the same formulas but starting with $D_V(s_V)$ instead of $D(s)$.

The set $\bar{\Omega}$ is defined in terms of the random variables N_x^n: $\forall x \in \mathbf{Z}^d$,

$$\Omega_x = \{ s \in \Omega \mid \exists n(x)\ \forall n \geqslant n(x), N_x^n = 0 \}$$

and

$$\Omega = \bigcap_{x \in \mathbf{Z}^d} \Omega_x \tag{28}$$

One can write $\bar{\Omega} = \bar{\Omega}_- \cup \bar{\Omega}_+$, according to the sign of the unique infinite subset of the lattice of constant sign. See the remark following Lemma 5 in ref. 1. The point of this definition of $\bar{\Omega}$ is of course that it is large enough in the sense that one can show that it has measure one and small enough to ensure the summability of the potentials that we shall define below. Roughly speaking, it means that around each point of the lattice the size of a cluster of bad configuration is of finite size. However, as we shall see later, it is possible to introduce another set $\Omega_{\log}$ that is still of measure 1 and such that the size of the bad cluster around each point is not only finite, but does not grow faster than some power of the logarithm of the distance of the point from the origin.

The relative energy of two configurations on the new lattice $\mathscr{L}$ in a volume V is obtained as the logarithm of the limit of the ratio of two partition functions of the original Ising model with constraints (for the deterministic transformations) or with the addition of interactions of finite range (given by the size of the boxes B_x) to the energy

$$\sum_{X \cap V \neq \varnothing} (\Phi_X(s^1_X) - \Phi_X(s^2_X)) = \log \lim_{\Lambda_2 \uparrow \mathscr{L}} \lim_{\Lambda_1 \uparrow \mathbf{Z}^d} \frac{Z^+_{\Lambda_1}(s^1_{\Lambda_1})}{Z^+_{\Lambda_1}(s^2_{\Lambda_2})} \tag{29}$$

with $s^1_x = s^2_x = \bar{s}_x \ \forall x \notin V$, $\bar{s} \in \bar{\Omega}_+$ and

$$Z^+_{\Lambda_1}(s_{\Lambda_2}) = \sum_{\sigma_{\Lambda_1}} \prod_{x \in \Lambda_2} T(s_x, \sigma_{B_x}) \exp -\beta H^I(\sigma_{\Lambda_1} \mid +) \tag{30}$$

Of course, we have a similar formula with $+$ replaced by $-$. The above formula (29) is obtained through an inductive representation, on each scale L^n, of the partition function (30) expressed in terms of polymers (contours). It follows then easily that the measures μ'_+ and μ'_- are weakly Gibbsian for the potentials Φ_X's (after one shows that the functions Φ_X's do not depend on the sign of the boundary condition on Λ_1). Alternatively, one can consider the following partition function:

$$Z^f_\Lambda(s_V) = \sum_{\sigma_\Lambda} \prod_{x \in V} T(s_x, \sigma_{B_x}) \exp -\beta H^I(\sigma_\Lambda \mid f) \tag{31}$$

with $\Lambda = \bigcup_{x \in V} B_x$ and following the inductive procedure of ref. 1, one obtains a representation analogous to the one in ref. 1 (see the appendix),

$$Z^f_\Lambda(s_V) = (\bar{T})^{|V|} \, e^{f^m_{f,\Lambda}(s_V)} \exp \sum_{X \subset V} \Phi^m_X(s_X) \, \tilde{Z}^m(s_V) \equiv (\bar{T})^{|V|} \, \hat{Z}^f_\Lambda(s_V) \tag{32}$$

with m such that $|[L^{-m}V]| \geqslant L^d$. In the following, we shall always consider the above relation with $m = m(V)$, where $m(V)$ is the largest m such that

$|[L^{-m}V]| \geqslant L^d$, obviously, one has $L^{m(V)d} = C(L)\,|V|$ and $m(V)$ goes to infinity with $V \uparrow \mathscr{L}$. The following properties of $\tilde{Z}^{m(V)}(s_V)$ are derived in the appendix:

$$\exp(-C(L,\beta)\,|V|) \leqslant \tilde{Z}^{m(V)}(s_V) \leqslant \exp(C(L,\beta)\,|V|) \tag{33}$$

and for $s \in \bar{\Omega}$

$$\tilde{Z}^{m(V)}(s_V) \to 1 \tag{34}$$

when V goes to infinity.

Let us now describe the functions Φ^m_X and $f^m_{f,\Lambda}$. For each connected X the potential Φ^m_X is made of two parts

$$\Phi^m_X(s_X) = \Phi^{m1}_X(s_X) + \Phi^{m2}_X(s_X) \tag{35}$$

with,

$$\Phi^{m1}_X(s_X) = \sum_{n=0}^{m} \sum_{D_i \in \mathscr{D}^n} \ln \rho^n(D_i)\,\chi(L^n \bar{D}_i = X) \tag{36}$$

and

$$\Phi^{m2}_X(s_X) = \sum_{n=0}^{m} \sum_{Y} \varphi^n(Y)\,\chi(L^n \bar{Y} = X) \tag{37}$$

with $\rho^n(D_i)$ and $\varphi^n(Y)$ such that

$$|\ln \rho^n(D_i)| \leqslant C(L)\,\beta L^{nd} \tag{38}$$

and

$$|\varphi^n(Y)| \leqslant \exp(-\beta L^{n(1-4\alpha)} L^{-2\alpha}\,|Y|) \tag{39}$$

moreover $\varphi^n(Y) = 0$ unless Y is connected. α is a positive parameter already used in the definition of $\mathscr{D}$. L and α are chosen as explained at the beginning of this subsection. The limits $\lim_{m\to\infty} \Phi^{m1}_X(s_X)$ and $\lim_{m\to\infty} \Phi^{m2}_X(s_X)$ exist and

$$\Phi_X(s_X) = \lim_{m\to\infty} \Phi^{m1}_X(s_X) + \lim_{m\to\infty} \Phi^{m2}_X(s_X) \tag{40}$$

This is exactly the same definition as the one given in ref. 1. Observe that it is obvious from the construction that Φ_X is translation invariant. Let us see now why the potential defined in (35) is only pointwise absolutely summable. The part of the sum involving Φ^2_X does not cause any problem because the bound (39) is sufficient to show that

$$\sum_{X \ni x} |\Phi^2_X(s_X)| \tag{41}$$

is uniformly (in the configurations s) bounded by a constant C_2. One can even prove more, namely

$$\sum_{X \ni x} \exp(d(X))^{1-4\alpha} |\Phi^{m2}_X(s_X)| \leqslant C'_2 \tag{42}$$

$\forall m$, where $d(X)$ denotes the diameter of X. The bound (38) is obviously not enough to prove the analogue for the part of the sum involving Φ^1_X. The way out of this trouble is to show that, for all $s \in \bar{\Omega}$.

$$\sum_{X \ni x} |\Phi^1_X(s_x)| \tag{43}$$

is bounded by a finite sum. Let, for $x \in \mathscr{L}$,

$$n(x, s) \equiv \max\{n \mid [L^{-n}x] \in \overline{D^n}(s)\} \tag{44}$$

Roughly speaking it gives the maximum scale $L^{n(x, s)}$ at which the image $[L^{-n}x]$ of x is "touched" by a component of D^n. On $\bar{\Omega}$, this always occurs on a finite scale, i.e. if $s \in \bar{\Omega}$, $n(x, s) < \infty$. Using this fact and (38), one gets that (43) is bounded by

$$C_1(x, s) \equiv C(L)\, \beta \sum_{n=0}^{n(x, s)} L^{nd} = C(L)\, \beta L^{dn(x, s)} \tag{45}$$

which is finite on $\bar{\Omega}$. $n_V(x, s)$ is defined similarly, with D^n_V replacing D^n in (44). We refer to the proof of Lemma 4 in ref. 1 for the proofs of all the above statements. Let us come to $f^m_{f, \Lambda}$,

$$f^m_{f, \Lambda}(s_V) = \sum_{n=0}^{m} \sum_{Y \subset [L^{-n}\Lambda]} \varphi^n(Y)\, \chi(L^n \bar{Y} \cap V^c \neq \varnothing)$$
$$+ \sum_{D_i \in \mathscr{D}^n_V} \ln \rho^n(D_i)\, \chi(L^n \bar{D}_i \cap V^c \neq \varnothing) \tag{46}$$

One can see that it collects some boundary terms because of the characteristic functions involved in the sum. We want to conclude this section by noting that, although the construction of the potential involve some parameters (L and α), the relative energy is obviously independent of those parameters, see (29).

3.2. A Class of Tempered Measures

We define here a class of tempered measures $\mathscr{M}$.

$$\mathscr{M} \equiv \left\{ \text{translation invariant measures } \mu \mid \exists \varepsilon < 0, \mu(N_x^n \neq 0) \leqslant \frac{C(\varepsilon)}{L^{nd(1+\varepsilon)}} \right\} \tag{47}$$

We shall need two basic properties of these tempered measures: $\forall \mu \in \mathscr{M}$,

$$\left| \mu\left(\sum_{X \ni 0} \Phi_X(\cdot) \right) \right| < \infty \tag{48}$$

and $\mu(\bar{\Omega}) = 1$. This last property is obvious because, using the definitions (28), we see that it is enough to show that $\mu(\Omega_x^c) = 0$ and this is proven by noticing that $\forall \mu \in \mathscr{M}$, one has $\mu(\Omega_x^c) \leqslant \lim_{N \to \infty} \sum_{n=N}^{\infty} \mu(N_x^n \neq 0) = 0$. Let us now prove (48).

$$\left| \mu\left(\sum_{X \ni 0} \Phi_X(\cdot) \right) \right| \leqslant \mu\left(\sum_{X \ni 0} |\Phi_X^1(\cdot)| \right) + \mu\left(\sum_{X \ni 0} |\Phi_X^2(\cdot)| \right) \tag{49}$$

The expectation value involving Φ_X^2 is obviously bounded because $\sum_{X \ni 0} |\Phi_X^2|$ is uniformly bounded. Using (45), we get,

$$\mu\left(\sum_{X \ni 0} |\Phi_X^1(\cdot)| \right) \leqslant C(L)\, \beta \mu(L^{n(0,\cdot)\, d}) \tag{50}$$

Now, $n(0, s) = N$ means by (27) that $\exists y$ such that $d(y, [L^{-N}0]) \leqslant 2$ and $N_y^N \neq 0$ and since there is a finite number of such y's, using the definition of $\mathscr{M}$, we get

$$(50) \leqslant C(L, \varepsilon)\, \beta \sum_{N=0}^{\infty} \frac{L^{Nd}}{L^{Nd(1+\varepsilon)}} < \infty \tag{51}$$

which is the desired result. It is easy to see that μ'_+ and μ'_- are in $\mathscr{M}$ using (4.44) in ref. 1.

4. RESULTS

We now introduce the set of boundary configurations $\Omega_{\log}$ for which one can show the existence of the thermodynamic limit of the pressure.

$$\Omega_{\log} \equiv \{ s \mid \exists a \text{ finite connected } \Lambda(s) \subset \mathbf{Z}^d, \forall x \notin \Lambda(s)\ C_1(x, s) \leqslant c_1 \beta (\log |x|)^{4d/\alpha} \} \tag{52}$$

where $C_1(x, s)$ is defined in (45) and gives, roughly speaking, the size of the "bad cluster" around the point x. The exponent $4d/\alpha$ on the log is put for convenience to match the probabilistic estimates of ref. 1. This set is a subset of $\bar{\Omega}$ and is also of measure 1 with respect to μ'_+ and μ'_-, if c_1 is chosen large enough. Obviously, if $s \in \Omega_{\log}$ then $\exists \Lambda(s)$ and $\exists c'_1$ such that $\forall x \notin \Lambda(s)$, $\sum_{X \ni x} |\Phi_X(s_X)| < c'_1 \beta (\log |x|)^{4d/\alpha}$. Then, our result is;

Theorem 1. Let Φ be the potential defined in (40) for which the measures μ'_+ and μ'_- are weakly Gibbsian, then for c_1 in (52) large enough, $\mu'_+(\Omega_{\log}) = \mu'_-(\Omega_{\log}) = 1$ and $\forall \mu \in \mathcal{M}$,

$$|\mu(A_\Phi)| < \infty \tag{53}$$

with,

$$A_\Phi = \sum_{X \ni 0} \frac{\Phi_X}{|X|} \tag{54}$$

$\forall \bar{s} \in \Omega_{\log}$ the limit of the pressure $p^{\bar{s}}$ exists, is finite, independent of the boundary conditions and proportional to the pressure of the original Ising model, $p^{\bar{s}} \equiv p = b^d p_{\text{Ising}}$. Moreover, $\forall \mu \in \mathcal{M}$,

$$-p \leqslant \mu(A_\Phi) + s(\mu) \tag{55}$$

and

$$-p = \mu'_+(A_\Phi) + s(\mu'_+) = \mu'_-(A_\Phi) + s(\mu'_-) \tag{56}$$

5. PROOF OF THE THEOREM

In the sequel, we shall denote by $\mu'(s_V \mid \bar{s}_{V^c})$ the conditional probabilities of the measures μ'_+ and μ'_- and we shall use the notation

μ' when an identity is true for both measures. Let us first show that $\mu'(\Omega_{\log}) = 1$. We shall prove that $\mu'(\Omega^c_{\log}) = 0$. By definition,

$$\Omega^c_{\log} = \bigcap_{m=0}^{\infty} \Omega^c_{\log, m} \tag{57}$$

with

$$\Omega^c_{\log, m} = \{s \mid \exists x \notin \Lambda_m, C_1(x, s) < c_1 \beta(\log |x|)^{4d/\alpha}\} \tag{58}$$

where Λ_m is a cube of side m and we have,

$$\Omega^c_{\log, m} \subset \bigcup_{x \notin \Lambda_m} \Omega^c_{\log}(x) \tag{59}$$

with,

$$\Omega^c_{\log}(x) = \{s \mid C_1(x, s) < c_1 \beta(\log |x|)^{4d/\alpha}\} \tag{60}$$

By the definition (45) of $C_1(x, s)$, this means that there is at least one site y such that $d(y, [L^{-n(x, s)}x]) \leqslant 2$ and $N_y^{n(x, s)} \neq 0$ with a lower bound on $n(x, s)$ given by

$$C(L)\, \beta L^{n(x, s)\, d} < c_1 \beta(\log |x|)^{4d/\alpha} \tag{61}$$

from which we get that $n(x, s)$ must be at least such that $L^{(n(x, s)\, \alpha)/4} > C(L) \log |x|$, using then the probabilistic estimates (4.44) and Lemma 6 in ref. 1, we get

$$\mu'(\Omega^c_{\log}(x)) \leqslant \exp(-c \sqrt{\beta}\, L^{\alpha n(x, s)/4}) \tag{62}$$

Thus, we finally obtain that

$$\mu'(\Omega_{\log, m}) \leqslant \sum_{x\, :\, |x| > m} \exp(-C(L) \sqrt{\beta} \log |x|) \tag{63}$$

So, if we choose $C(L) \sqrt{\beta}$ large enough, by taking c_1 large enough, the result $\mu'(\Omega_{\log}) = 1$ follows.

Let us now prove the remaining part of the theorem, $|\mu(A_\Phi)| < \infty$ follows obviously from (48) in Subsection 3.2. Then, notice, following ref. 12, that for $\bar{s} \in \Omega_{\log}$ and $\forall \mu \in \mathscr{M}$, the following identity holds true:

$$\frac{1}{|V|} S_V(\mu \mid \mu'_V(\cdot \mid \bar{s}_{V^c})) = \frac{1}{|V|} S_V(\mu) + \frac{1}{|V|} \mu(H_V(\cdot \mid \bar{s}_{V^c})) + \frac{1}{|V|} \log Z_V(\bar{s}_{V^c}) \tag{64}$$

indeed, we have that

$$\sum_{s_V} \mu(s_V) \log \frac{\mu(s_V)}{\mu'(s_V \mid \bar{s}_{V^c})} = \sum_{s_V} \mu(s_V) \log \mu(s_V) - \sum_{s_V} \mu(s_V) \log \mu'(s_V \mid \bar{s}_{V^c}) \quad (65)$$

$$= S_V(\mu) + \mu(H_V(\cdot \mid \bar{s}_{V^c})) + \log Z_V(\bar{s}_{V^c}) \quad (66)$$

where we used the DLR equations (10) to obtain the last line and the fact that $\Omega_{\log} \subset \bar{\Omega}$.

Observe that the limit of the first term in the right-hand side of the equality (64) exists and is finite because it is the entropy density of the measure μ, as remarked after the definition (18). Then, the proof of the theorem is done if we take the limit V going to infinity in (64) and if we prove the three following propositions.

Proposition 1. $\forall \bar{s} \in \Omega_{\log}$,

$$\lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} S_V(\mu' \mid \mu'_V(\cdot \mid \bar{s}_{V^c})) = 0 \quad (67)$$

Proposition 2. $\forall \mu \in \mathscr{M}$ and $\forall \bar{s} \in \Omega_{\log}$,

$$\lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} \mu(H_V(\cdot \mid \bar{s}_{V^c})) = \mu(A_{\Phi}) < \infty \quad (68)$$

and

Proposition 3. $\forall \bar{s} \in \Omega_{\log}$,

$$p^{\bar{s}} = \lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} \log \sum_{s_V} \exp \sum_{X \cap V \neq \varnothing} \Phi_X(s_{X \cap V} \vee \bar{s}_{X \cap V^c}) = b^d p_{\text{Ising}} \quad (69)$$

Indeed, Proposition 1 and Proposition 2 proves the variational equality (56) in the Theorem if $\mu = \mu'$, and show indirectly the existence of the pressure and its independence of the boundary conditions. The fact that the relative entropy is always positive and Proposition 2 proves the inequality. Proposition 3 shows that the free energy that we obtain is the free energy of the Ising model modulo a scale factor. It also provides a more direct proof of the existence of the pressure.

Proof of Proposition 1. In order to prove this proposition we need the following lemma whose proof is given in the Appendix:

Lemma 1.

$$\mu'(s_V \mid \bar{s}_{V^c}) = \sum_{\sigma_\Lambda} \nu(\sigma_\Lambda \mid \bar{s}_{V^c})\, T(\sigma_\Lambda, s_V) \tag{70}$$

where

$$\nu(\sigma_\Lambda \mid \bar{s}_{V^c}) = \sum_{\hat{\sigma}_{\partial\Lambda}} \frac{\exp - H^I(\sigma_\Lambda \mid \hat{\sigma}_{\partial\Lambda})}{Z^I_\Lambda(\hat{\sigma}_{\partial\Lambda})}\, \tilde{\mu}^{\bar{s}_{V^c}}_{\partial\Lambda}(\hat{\sigma}_{\partial\Lambda})$$

for each $\bar{s}_{V^c}$, $\tilde{\mu}^{\bar{s}_{V^c}}_{\partial\Lambda}$ is a probability measure on $\Omega_{\partial\Lambda}$ and $\Lambda = \bigcup_{x\in V} B_x$.

Using then (14), the positivity of the relative entropy and (16), we may conclude the proof of Proposition 1.

Proof of Proposition 2. First, it is easy to see that one can write $\Omega_{\log} = \bar{\Omega}_{+\log} \cup \Omega_{-\log}$ with $\Omega_{\pm\log} = \Omega_\pm \cap \Omega_{\log}$. In the following we shall alwaysassume that $s \in \Omega_{+\log}$, if s was in $\Omega_{-\log}$, the only thing to change would be the sign on the partition function in (75). Let $\mu \in \mathscr{M}$ and $\bar{s} \in \Omega_{\log}$, write,

$$\begin{aligned}
&\frac{1}{|V|}\mu(H_V(\cdot \mid \bar{s}_{V^c}))\\
&\quad = \frac{1}{|V|}\mu\left(\sum_{X\cap V\neq\varnothing} \Phi_X(\cdot \vee \bar{s}_{X\cap V^c}) - \sum_{X\cap V\neq\varnothing} \Phi_X(\bar{s}_X)\right)\\
&\qquad + \frac{1}{|V|}\sum_{X\cap V\neq\varnothing} \Phi_X(\bar{s}_X)
\end{aligned} \tag{71}$$

We do this in order to be able to use (29) which expresses the relative energy in the volume V in terms of the logarithm of a ratio of partition functions of the original Ising model with local interactions (or constraints). This will serve to eliminate the dependence of the first two terms on the configuration outside the volume V. But let us look first at the third term in the above expression,

$$\frac{1}{|V|}\sum_{X\cap V\neq\varnothing} \Phi_X(\bar{s}_X) = \frac{1}{|V|}\sum_{y\in V}\sum_{X\ni y} \frac{1}{|X\cap V|}\Phi_X(\bar{s}_X) \tag{72}$$

$$\begin{aligned}
&= \frac{1}{|V|}\sum_{y\in V}\sum_{\substack{X\ni y\\ X\subset V}} \frac{1}{|X|}\Phi_X(\bar{s}_X)\\
&\quad + \frac{1}{|V|}\sum_{y\in V}\sum_{\substack{X\ni y\\ X\cap V^c\neq\varnothing}} \frac{1}{|X\cap V|}\Phi_X(\bar{s}_X)
\end{aligned} \tag{73}$$

Using the following Lemma, one sees that the second term of this expression, which groups the terms involving the configuration outside the volume V, goes to zero.

Lemma 2. $\forall \bar{s} \in \Omega_{\log}$,

$$\lim_{V\uparrow\mathscr{L}} \frac{1}{|V|} \sum_{y\in V} \sum_{\substack{X\ni y\\ X\cap V^c\neq\varnothing}} \frac{1}{|X\cap V|} |\Phi_X(\bar{s}_X)| = 0 \tag{74}$$

Using (29), the first two terms of (71) can be reexpressed as

$$\frac{1}{|V|}\mu\left(\lim_{\Lambda_2\uparrow\mathscr{L}} \lim_{\Lambda_1\uparrow\mathbf{Z}^d} \log \frac{Z^+_{\Lambda_1}(\cdot \vee \bar{s}_{V^c\cap\Lambda_2})}{Z^+_{\Lambda_1}(\bar{s}_{\Lambda_2})}\right) \tag{75}$$

With the help of (30) and (17), one obtains the following lower and upper bounds on the ratio of the partition functions

$$e^{-4\beta\,|\partial\Lambda|} \frac{Z^f_\Lambda(s_V)}{Z^f_\Lambda(\bar{s}_V)} \leqslant \frac{Z^+_{\Lambda_1}(s_V \vee \bar{s}_{V^c\cap\Lambda_2})}{Z^+_{\Lambda_1}(\bar{s}_V \vee \bar{s}_{V^c\cap\Lambda_2})} \leqslant \frac{Z^f_\Lambda(s_V)}{Z^f_\Lambda(\bar{s}_V)}\, e^{4\beta\,|\partial\Lambda|} \tag{76}$$

with $\Lambda = \bigcup_{x\in V} B_x$ and $\partial\Lambda = \{x\in\Lambda \mid d(x,\Lambda^c) = 1\}$. Taking then the limit on V in (75), we obtain

$$\lim_{V\uparrow\mathscr{L}} \frac{1}{|V|}\mu\left(\lim_{\Lambda_2\uparrow\mathscr{L}} \lim_{\Lambda_1\uparrow\mathbf{Z}^d} \log \frac{Z^+_{\Lambda_1}(\cdot \vee \bar{s}_{V^c\cap\Lambda_2})}{Z^+_{\Lambda_1}(\bar{s}_{\Lambda_2})}\right) = \lim_{V\uparrow\mathscr{L}} \frac{1}{|V|}\mu\left(\log\frac{\hat{Z}^f_\Lambda(\cdot)}{\hat{Z}^f_\Lambda(\bar{s}_V)}\right) \tag{77}$$

The following Lemma shows that the second and the third term cancel.

Lemma 3. $\forall \bar{s} \in \Omega_{\log}$,

$$\lim_{V\uparrow\mathscr{L}} \frac{1}{|V|} \log \hat{Z}^f_\Lambda(\bar{s}) = \lim_{V\uparrow\mathscr{L}} \frac{1}{|V|} \sum_{y\in V} \sum_{\substack{X\ni y\\ X\subset V}} \frac{1}{|X|} \Phi_X(\bar{s}_X) \tag{79}$$

and the last Lemma of this section concludes the proof of Proposition 2.

Lemma 4. $\forall \mu \in \mathscr{M}$,

$$\lim_{V\uparrow\mathscr{L}} \frac{1}{|V|} \mu(\log \hat{Z}^f_\Lambda(\cdot)) = \mu(A_\Phi) \tag{80}$$

These last two lemmas are a consequence of the representation (32) establishing the relationship between the Ising partition function with constraints and our potentials Φ_X. Note that (80) is not a consequence of (79) because $\mu(\Omega_{\log})=1$ does not hold for all $\mu \in \mathcal{M}$. Note also that the existence of the left-hand side of (79) could be obtained μ' almost surely by standard arguments in disordered systems (ergodicity), since the spins $\{s_x \mid x \in V\}$ can be seen as random fields on the Ising model.

Proof of Proposition 3.

$$p^{\bar{s}} = \lim_{V\uparrow \mathcal{L}} \frac{1}{|V|} \log \sum_{s_V} \exp \sum_{X\cap V\neq\varnothing} (\Phi_X(s_{X\cap V} \vee \bar{s}_{X\cap V^c}) - \Phi_X(\bar{s}_X)) + \lim_{V\uparrow \mathcal{L}} \frac{1}{|V|} \sum_{X\cap V\neq\varnothing} \Phi_X(\bar{s}_X) \tag{81}$$

Using then (29) and following the steps leading from (75) to (77), we obtain,

$$p^{\bar{s}} = \lim_{V\uparrow \mathcal{L}} \frac{1}{|V|} \log \sum_{s_V} \frac{Z_\Lambda^f(s_V)}{Z_\Lambda^f(\bar{s}_V)} + \lim_{V\uparrow \mathcal{L}} \frac{1}{|V|} \sum_{X\cap V\neq\varnothing} \Phi_X(\bar{s}_X) \tag{82}$$

Now, by (32), (74) in Lemma 2 and (79) in Lemma 3, we see that

$$p^{\bar{s}} = \lim_{V\uparrow \mathcal{L}} \frac{1}{|V|} \log \sum_{s_V} Z_\Lambda^f(s_V) \tag{83}$$

Clearly, by (2) and (31), $\sum_{s_V} Z_\Lambda^f(s_V)$ is the partition function of the original Ising model in the volume Λ and because $|\Lambda| = b^d\,|V|$, we finally get,

$$p^{\bar{s}} = b^d p_{\text{Ising}} \tag{84}$$

This concludes the proof of Proposition 3.

APPENDIX

A.1. Proof of Lemma 1

This follows mainly from algebraic manipulations. We may construct $\mu'(s_V \mid \bar{s}_{V^c})$ in the following way:

$$\mu'(S_V \mid \bar{s}_{V^c}) = \lim_n \frac{\mu'(s_V \vee \bar{s}_{V^c \cap V_n})}{\mu'(\bar{s}_{V^c \cap V_n})} \tag{85}$$

for an increasing sequence of cubes (V_n), $V_n \subset V_{n+1}$. In the following, we keep the notation $\Lambda = \bigcup_{x \in V} B_x$ and $\Lambda_n = \bigcup_{x \in V_n} B_x$.

$$\mu'(s_V \vee \bar{s}_{V^c \cap V_n})$$

$$= \sum_{\sigma_\Lambda} \sum_{\hat{\sigma}_{\Lambda^c \cap \Lambda_n}} \mu(\sigma_\Lambda \vee \hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \prod_{x \in V} T_x(\sigma_{B_x}, s_x) \prod_{x \in V^c \cap V_n} T_x(\hat{\sigma}_{B_x}, \bar{s}_x) \tag{86}$$

$$= \sum_{\sigma_\Lambda} \sum_{\hat{\sigma}_{\Lambda^c \cap \Lambda_n}} \mu(\sigma_\Lambda \mid \hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \, \mu(\hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \times \prod_{x \in V} T_x(\sigma_{B_x}, s_x) \prod_{x \in V^c \cap V_n} T_x(\hat{\sigma}_{B_x}, \bar{s}_x) \tag{87}$$

$$= \sum_{\sigma_\Lambda} \prod_{x \in V} T_x(\sigma_{B_x}, s_x) \sum_{\hat{\sigma}_{\Lambda^c \cap \Lambda_n}} \frac{\exp -H^I_\Lambda(\sigma_\Lambda \mid \hat{\sigma}_{\partial\Lambda})}{Z^I_\Lambda(\hat{\sigma}_{\partial\Lambda})} \mu(\hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \times \prod_{x \in V^c \cap V_n} T_x(\hat{\sigma}_{B_x}, \bar{s}_x) \tag{88}$$

Thus, we obtain that $\mu'(s_V \mid \bar{s}_{V^c})$

$$= \lim_n \sum_{\sigma_\Lambda} \prod_{x \in V} T_x(\sigma_{B_x}, s_x) \sum_{\hat{\sigma}_{\Lambda^c \cap \Lambda_n}} \frac{\exp -H^I_\Lambda(\sigma_\Lambda \mid \hat{\sigma}_{\partial\Lambda})}{Z^I_\Lambda(\hat{\sigma}_{\partial\Lambda})} \times \frac{\mu(\hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \prod_{x \in V^c \cap V_n} T_x(\hat{\sigma}_{B_x}, \bar{s}_x)}{\mu'(\bar{s}_{V^c \cap V_n})} \tag{89}$$

Next, we define

$$\tilde{\mu}^{\bar{s}_{V^c \cap V_n}}_{\Lambda^c \cap \Lambda_n}(\hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \equiv \frac{\mu(\hat{\sigma}_{\Lambda^c \cap \Lambda_n}) \prod_{x \in V^c \cap V_n} T_x(\hat{\sigma}_{B_x}, \bar{s}_x)}{\mu'(\bar{s}_{V^c \cap V_n})} \tag{90}$$

It is easily checked that it is a probability distribution on $\Omega_{\Lambda^c \cap \Lambda_n}$. We can then take the restriction of this measure to $\Omega_{\partial\Lambda}$ and write, because of the form of H^I_Λ;

$$\mu'(s_V \mid \bar{s}_{V^c}) = \lim_n \sum_{\sigma_\Lambda} \prod_{x \in V} T_x(\sigma_{B_x}, s_x) \sum_{\hat{\sigma}_{\partial\Lambda}} \frac{\exp -H^I_\Lambda(\sigma_\Lambda \mid \hat{\sigma}_{\partial\Lambda})}{Z^I_\Lambda(\hat{\sigma}_{\partial\Lambda})} \cdot \tilde{\mu}^{\bar{s}_{V^c \cap V_n}}_{\partial\Lambda, n}(\hat{\sigma}_{\partial\Lambda}) \tag{91}$$

By compactness, $\lim_m \mu^{\bar{s}_{V^c \cap V_m}}_{\partial\Lambda, m}(\cdot)$ exists for a subsequence and is a probability measure on $\Omega_{\partial\Lambda}$, and we finally obtain

$$\mu'(s_V \mid \bar{s}_{V^c}) = \sum_{\sigma_\Lambda} \prod_{x \in V} T_x(\sigma_{B_x}, s_x) \sum_{\hat{\sigma}_{\partial\Lambda}} \frac{\exp -H^I_\Lambda(\sigma_\Lambda \mid \hat{\sigma}_{\partial\Lambda})}{Z^I_\Lambda(\hat{\sigma}_{\partial\Lambda})} \cdot \tilde{\mu}^{\bar{s}_{V^c}}_{\partial\Lambda}(\hat{\sigma}_{\partial\Lambda}) \tag{92}$$

A.2. Proof of Lemma 2

For the sake of simplicity take the V's to be a sequence of cube of side $2l$ (then $|V| = (2l+1)^d$) centered around the origin. Let $C(\Lambda(\bar{s})) = \sup_{x \in \Lambda(\bar{s})} C_1(x, \bar{s})$, we shall always assume that V is large enough so that $c_1 \beta(\log |l|)^{4d/\alpha} \geqslant C(\Lambda(\bar{s}))$ holds. We may decompose the expression in (74) in the following way:

$$\frac{1}{|V|} \sum_{\substack{y \in V \\ d(y, V^c) > l^{1/2}}} \sum_{\substack{X \ni y \\ X \cap V^c \neq \emptyset}} \frac{1}{|X \cap V|} |\Phi_X(\bar{s}_X)| + \frac{1}{|V|} \sum_{\substack{y \in V \\ d(y, V^c) \leqslant l^{1/2}}} \sum_{\substack{X \ni y \\ X \cap V^c \neq \emptyset}} \frac{1}{|X \cap V|} |\Phi_X(\bar{s}_X)| \tag{93}$$

Now, using the definition of $\Omega_{\log}$, V and Λ, we further bound the first term above, because all the terms involved in the sum must have $|X \cap V| > l^{1/2}$ and $|y| < l$, and obtain:

$$\frac{1}{|V|} \sum_{\substack{y \in V \\ d(y, V^c) > l^{1/2}}} \sum_{\substack{X \ni y \\ X \cap V^c \neq \emptyset}} \frac{1}{|X \cap V|} |\Phi_X(\bar{s}_X)| \leqslant \frac{c_1' \beta(\log l)^{4d/\alpha}}{l^{1/2}} \tag{94}$$

so that the first term of (93) goes to zero. Let us now show that its second term does so too. It can be bounded in the same way and because there are at most $cl^{1/2}(2l+1)^{d-1}$ points $y \in V$ such, that $d(y, V^c) \leqslant l^{1/2}$, we get:

$$\frac{1}{|V|} \sum_{y \in \Lambda} \sum_{X \ni y} \frac{1}{|X \cap V|} |\Phi_X(\bar{s}_X)| \leqslant \frac{cl^{1/2} c_1' \beta(\log l)^{4d/\alpha}}{2l+1} \tag{95}$$

so, the proof is finished by letting l going to infinity.

A.3. Proof of Lemma 3

By the representation (32),

$$\log \hat{Z}_\Lambda(\bar{s}_V) = f_{f,\Lambda}^{m(V)}(\bar{s}_V) + \sum_{X \subset V} \Phi_X^{m(V)}(\bar{s}_X) + \log \tilde{Z}^{m(V)}(\bar{s}_V) \tag{96}$$

and using (34), it is clear that the limit of the third term divided by $|V|$ goes to zero. Let us then prove that $\forall \bar{s} \in \Omega_{\log}$,

$$\lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} f_{f,\Lambda}^{m(V)}(\bar{s}_V) = 0 \tag{97}$$

As $f^{m(V)}_{f,\Lambda}(\bar{s}_V)$ regroups some "boundary terms," we shall reduce the proof of this fact to the proof of Lemma 2.

Define

$$\tilde{\Phi}^{m1}_X(s_X) = \sum_{n=0}^{m} \sum_{D_i \in \mathscr{D}^n_V} \ln \rho^n(D_i)\, \chi(L^n \bar{D}_i = X) \tag{98}$$

$$\tilde{\Phi}^{m2}_X(s_X) = \sum_{n=0}^{m} \sum_{Y \subset [L^{-m}\Lambda]} \varphi^n(Y)\, \chi(L^n \bar{Y} = X) \tag{99}$$

and $\tilde{\Phi}^m_X = \tilde{\Phi}^{m1}_X + \tilde{\Phi}^{m2}_X$. We define $\tilde{\Phi}_X$ to emphasize the difference with respect to Φ_X where the sum runs over connected components of $\mathscr{D}^n$. Observe that $\tilde{\Phi}^{m1}_X = 0$ and $\tilde{\Phi}^{m2}_X = 0$ if $X \cap V = \varnothing$ because $\mathscr{D}^n_V \cap [L^{-n}V] \neq \varnothing$ and the sum in (99) runs over $Y \subset [L^{-m}\Lambda]$. Then, we can writte, using (46),

$$f^{m(V)}_{f,\Lambda}(\bar{s}_V) = \sum_{X \cap V^c \neq \varnothing} \tilde{\Phi}^{m(V)}_X(\bar{s}_X) = \sum_{\substack{X \cap V^c \neq \varnothing \\ X \cap V \neq \varnothing}} \tilde{\Phi}^{m(V)}_X(\bar{s}_X) \tag{100}$$

$$f^{m(V)}_{f,\Lambda}(\bar{s}_V) = \sum_{y \in V} \sum_{\substack{X \cap V^c \neq \varnothing \\ X \ni y}} \frac{1}{|X \cap V|} \tilde{\Phi}^{m(V)}_X(\bar{s}_X) \tag{101}$$

Now, we have that

$$\sum_{X \ni y} |\tilde{\Phi}^{m1}_X(\bar{s})| \leqslant C_1(y, \bar{s}) \tag{102}$$

$\forall m$, because using the bound (38), this sum can be bounded in the same way than (43) by $C(L)\, \beta L^{n(y,\bar{s})\, d}$ and $n_V(y, \bar{s}) \leqslant n(y, \bar{s})$ (because $D^n_V \subset D^n$). Using then the bound (39) on φ^n, we get that $\forall \bar{s} \in \Omega_{\log}$, $\exists \Lambda(\bar{s})$ such that $\forall y \notin \Lambda(\bar{s})$,

$$\sum_{X \ni y} |\tilde{\Phi}^m_X(\bar{s})| \leqslant c'_1 \beta \log |y| \tag{103}$$

$\forall m$. Thus, the proof of $\lim_{V \uparrow \mathscr{L}} (1/|V|)\, f^{m(V)}_{f,\Lambda}(\bar{s}_V) = 0$ boils down to the arguments used in the proof of Lemma 2.

We are now left with proving that

$$\lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} \sum_{X \subset V} \Phi^{m(V)}_X(\bar{s}_X) = \lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} \sum_{y \in V} \sum_{\substack{X \ni y \\ X \subset V}} \frac{1}{|X|} \Phi_X(\bar{s}_X) \tag{104}$$

The main difficulty in doing this lies in the fact that the objects over which we sum *and* the sum itself depend on V, hence, we need the following

Lemma 5. $\forall s \in \Omega_{\log}$, $\exists \bar{V}$ such that $\forall V \supset \bar{V}$, $\forall y \in V$ and $\forall X \ni y$,

$$\Phi_X^{m(V)\,1}(s_X) = \Phi_X^1(s_X) \tag{105}$$

and, $\forall s$, $\forall y$

$$\sum_{X \ni y} |\Phi_X^2(s_X) - \Phi_X^{m(V)2}(s_X)| \leqslant c \exp -cL^{m(V)(1-4\alpha)} \tag{106}$$

It is then clear that we may obtain the desired result.

A.4. Proof of Lemma 4

We have,

$$\mu(\log \hat{Z}_\Lambda(\cdot)) = \mu(f_{f,\Lambda}^{m(V)}(\cdot)) + \mu\left(\sum_{X \subset V} \Phi_X^{m(V)}(\cdot)\right) + \mu(\log \tilde{Z}^{m(V)}(\cdot)) \tag{107}$$

using (34) and the fact that $\forall \mu \in \mathscr{M}$, $\mu(\bar{\Omega}) = 1$, one sees that $\tilde{Z}^{m(V)}(s_V)$ converges pointwise μ-a.s. to 1, which, together with the bounds (33) and Lebesgue dominated convergence theorem, implies that

$$\lim_{V \uparrow \mathscr{L}} \mu\left(\frac{1}{|V|} \log \tilde{Z}^{m(V)}(\cdot)\right) = 0 \tag{108}$$

Let us now show that

$$\lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} |\mu(f_{f,\Lambda}^{m(V)}(\cdot))| = \lim_{V \uparrow \mathscr{L}} \frac{1}{|V|} \mu\left(\sum_{y \in V} \sum_{\substack{X \cap V^c \neq \varnothing \\ X \ni y}} \frac{1}{|X \cap V|} |\tilde{\Phi}_X^{m(V)}(\cdot)|\right) = 0 \tag{109}$$

To see this, observe that

$$|\tilde{\Phi}_X^{m(V)\,1}(s_X)| \leqslant \sum_{n=0}^{\infty} \sum_{D_i \in \mathscr{D}_V^n} |\ln \rho^n(D_i)|\, \chi(L^n \bar{D}_i = X) = \tilde{C}_X(s_X) \tag{110}$$

and following the proof of $\mu(\sum_{X \ni 0} \Phi_X(\cdot)) < \infty$, one sees that

$$\mu\left(\sum_{X \ni y} \tilde{C}_X(\cdot)\right) \leqslant C(L)\, \beta\mu(L^{n_V(y,\cdot)\,d}) \leqslant \mu(L^{n(y,\cdot)\,d}) \leqslant c < \infty \tag{111}$$

independently of y (by translation invariance of μ and n), where the second inequality follows from the fact that $D^n_V \subset D^n$. Thus because $\tilde{\Phi}^{m2}_X$ is uniformly absolutely summable (see Subsection 3.1), $|\tilde{\Phi}^m_X(s_X)|$ is itself bounded by a function C_X such that $\mu(\sum_{X \ni y} C_X(\cdot)) \leqslant c < \infty$. Using this fact, we obtain,

$$\mu\left(\frac{1}{|V|}\sum_{y\in V}\sum_{\substack{X\ni y\\ X\cap V^c\neq\varnothing}}\frac{1}{|X\cap V|}\tilde{\Phi}^{m(V)}_X(\cdot)\right)\leqslant\frac{1}{|V|}\sum_{y\in V}\sum_{\substack{X\ni y\\ X\cap V^c\neq\varnothing}}\bar{C}_X \tag{112}$$

with $\bar{C}_X = \mu(C_X(\cdot))$. This expression can be further bounded by

$$\frac{1}{|V|}\sum_{y\in V}\sum_{\substack{X\ni y\\ X\cap(W+y)^c\neq\varnothing}}\bar{C}_X+\frac{1}{|V|}\sum_{\substack{y\in V\\ (W+y)\cap V^c\neq\varnothing}}\sum_{X\ni y}\bar{C}_X \tag{113}$$

for any connected subset W of $\mathscr{L}$, where $W+y$ denotes the translate of W by y. The second term goes to zero as V goes to infinity and the first one can be made aribtrarily small by suitably chosing W.

Finally, let us now prove that

$$\lim_{V\uparrow\mathscr{L}}\frac{1}{|V|}\mu\left(\sum_{X\subset V}\Phi^{m(V)}_X(\cdot)\right)=\mu(A_\Phi) \tag{114}$$

Using translation invariance, write,

$$\begin{aligned}&\lim_{V\uparrow\mathscr{L}}\frac{1}{|V|}\mu\left(\sum_{X\subset V}\Phi^{m(V)}_X(\cdot)\right)\\ &\quad=\lim_{V\uparrow\mathscr{L}}\mu\left(\sum_{X\ni 0}\frac{1}{|X|}\Phi^{m(V)}_X(\cdot)\right)-\lim_{V\uparrow\mathscr{L}}\mu\left(\frac{1}{|V|}\sum_{y\in V}\sum_{\substack{X\ni y\\ X\cap V^c\neq\varnothing}}\frac{1}{|X|}\Phi^{m(V)}_X(\cdot)\right)\end{aligned} \tag{115}$$

The second term in the right-hand side is zero exactly for the same reason that (109) holds (replace everywhere D^n_V with D^n and the corresponding quantitites depending on it). The first one is $\mu(A_\Phi)$. This statement follows if we show

$$\lim_{m\to\infty}\mu\left(\sum_{X\ni 0}\frac{1}{|X|}\Phi^m_X(\cdot)\right)=\mu\left(\sum_{X\ni 0}\frac{1}{|X|}\Phi_X(\cdot)\right) \tag{116}$$

First, observe that $\lim_{m\to\infty}\sum_{X\ni 0}(1/|X|)\,\Phi^m_X(s_X)=\sum_{X\ni 0}(1/|X|)\,\Phi_X(s_X)$, μ-a.s., by dominated convergence (for the series), which follows from the proof of the summability of the potential explained in Subsection 3.1 (each $|\Phi^m_X|$ is bounded by C_X defined as above but with D^n instead of D^n_V) and $\lim_{m\to\infty} \Phi^m_X(s_X) = \Phi_X(s_X)$. Then, applying again dominated convergence using $|\mu(A_\Phi)| < \infty$ and the fact that $|\sum_{X\ni 0} (1/|X|)\, \Phi^m_X(s_X)|$ is bounded independently of m by a function that is μ-summable (e.g. $\sum_{X\in 0} C_X$), we conclude the proof of (116).

A.5. Proof of Lemma 5

Again, let $C(\Lambda(\bar{s})) = \sup_{x\in\Lambda(\bar{s})} C_1(x, \bar{s})$ and assume the V's to be a sequence of cube of side l and that V is large enough so that $c_1\beta(\log|l|)^{4d/\alpha} \geqslant C(\Lambda(\bar{s}))$. Let us prove first (105). First, observe that $\forall s\in\bar{\Omega}$, $\forall X\ni y$ one has

$$\Phi^{m1}_X(s_X) = \Phi^1_X(s_X) \tag{117}$$

$\forall m > n(y, s)$, because if it is not the case, $\exists m > n(y, s)$ and $D_i \in \mathscr{D}^m$ such that $[L^{-m}y] \in \bar{D}_i$, by (36). Using then the definition of $\Omega_{\log}$ and (45), one sees that $\forall \bar{s}\in\Omega_{\log}$, $n(x, s)$ is such that $L^{n(x,s)\,d} \leqslant C(L)(\log|y|)^{4d/\alpha}$. On the other hand, $L^{m(V)\,d} = C(L)\,|V| = C(L)\, l^d$, so that if we take V large enough, we haven $\forall y\in V(|y| < l)$, $m(V) > n(y, \bar{s})$ and thus, $\forall X\ni y$,

$$\Phi^{m(V)\,1}_X(s_X) = \Phi^1_X(s_X) \tag{118}$$

Let us now prove (106). Let $\bar{m}(X)$ be the largest m such that $\exists Y$ such that $L^m\bar{Y} = X$, obviously $\bar{m}(X)$ is such that $L^{\bar{m}(X)} \leqslant Cd(X)$, thus, using (37), one obtains that $\forall X$ such that $d(X) < cL^{m(V)}$,

$$\Phi^2_X(s_X) = \Phi^{m(V)\,2}_X(s_X) \tag{119}$$

and thus,

$$\sum_{X\ni y} |\Phi^2_X(s_X) - \Phi^{m(V)\,2}_X(s_X)| \leqslant \sum_{\substack{X\ni y \\ d(X) > cL^{m(V)}}} |\Phi^2_X(s_X)| + |\Phi^{m(V)\,2}_X(s_X)| \tag{120}$$

and using (42), we may bound this expression by $2C_2' \exp -cL^{m(V)(1-4\alpha)}$, which concludes the proof of Lemma 3.

A.6. Proof of Various Statements in Subsection 3.1

We refer to ref. 1 for all the relevant definitions.

Representation (32). Z^f_Λ is given by the sum (3.12) in ref. 1 if we take $\Lambda_2 = V$ and $\Lambda_1 = \Lambda$. Besides, in the case of free boundary conditions the signs of the contours are automatically compatible with it. Applying the results of Proposition 3 in ref. 1 (and the construction in the proof of it) to this particular case, (3.25) in ref. 1 becomes $\forall m$ such that $|[L^{-m}V]| > L^d$,

$$Z^f_\Lambda(s_V) = (\bar{T})^{|V|} e^{f^m_{f,\Lambda}(s_V)} \exp \sum_{X \subset V} \Phi^m_X(s_X) \tilde{Z}^m(s_V) \tag{121}$$

$\tilde{Z}^m(s_V)$ is given by (3.26) in ref. 1 and $f^m_{f,\Lambda}(s_V)$ is defined inductively by (4.20) in ref. 1. The potentials Φ^m_X are also defined inductively by (4.18) in ref. 1. In (36) we write the potential directly in terms of components of $\mathscr{D}^m$ (and not in terms of components of $\mathscr{D}^m_V$) because all the X's that we consider are included in V and because it is easy to see inductively that "D_i is connected component of $\mathscr{D}^m_V$ such that $L^m\bar{D}_i \subset V$" is equivalent to "D_i is connected component of $\mathscr{D}^m$ such that $L^m\bar{D}_i \subset V$." Similarly, in (37) we drop the superscript $+$ on φ^n in the definition (4.18) in ref. 1 because it is shown to be independent of the boundary conditions in Lemma 4 of ref. 1.

Properties (33) and (34). The proof of (34) is identical to the proof of (4.34) in ref. 1, once we prove that for $s \in \bar{\Omega}$ and for n large enough, $D^n_V = \varnothing$ which follows from the following observations, $D^n_V \subset D^n \cap [L^{-n}V]$, $[L^{-m(V)}V] \subset L^2\{0\}$ (where $L^2\{0\}$ is the cube of size L^2 around the origin) and if $s \in \bar{\Omega}$, $\exists n$ such that $\forall m > n$, $D^m(s) \cap L^2\{0\} = \varnothing$. The upper bound in (33) follows from the representation (3.26) in ref. 1. Observe that the number of families of contours in this equation is bounded by a constant $C(L)$ because $[L^{-m(V)}V] \subset L^2\{0\}$. Using then the fact that $L^{m(V)\,d} = C(L)\,|V|$, the bounds (3.29), (3.31) and Lemma 6 in ref. 1 the claim follows. The lower bound follows easily from the fact that $D^{m(V)}_V$ is obviously a family of $s^{m(V)}$-compatible contours, the bounds of the Lemma 6 and (3.29) in ref. 1, the positivity of the weights of the contours and again the fact that $L^{m(V)\,d} = C(L)\,|V|$.

ACKNOWLEDGMENTS

I would like to thank J. Bricmont for his guidance, discussions and numerous advises in organizing this paper. It is also a pleasure to thank A. van Enter, S. Focant, C. Maes, F. Redig and A. Van Moffaert for interesting discussions.

REFERENCES

1. J. Bricmont, A. Kupiainen, and R. Lefevere, Renormalization Group pathologies and the definition of Gibbs states, *Commun. Math. Phys.* **194**:359–388 (1998).

2. R. L. Dobrushin, Lecture given at the workshop "Probability and Physics," Renkum, Holland, 28 August–1 September (1995).
3. C. Domb and M. S. Green, Eds., *Phase Transitions and Critical Phenomena*, Vol. 6 (Academic Press, New York, 1976).
4. R. Fernández and C.-Ed. Pfister, Global specifications and non-quasilocality of projections of Gibbs measures, EPFL preprint (1996).
5. H.-O. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter Studies in Mathematics, Vol. 9 ((Walter de Gruyter, Berlin/New York, 1988).
6. R. B. Griffiths, Nonanalytic behavior above the critical point in a random Ising ferromagnet, *Phys. Rev. Lett.* **23**:17 (1969).
7. R. B. Griffith and P. A. Pearce, Position-space renormalization-group transformations: Some proofs and some problems, *Phys. Rev. Lett.* **41**:917–920 (1978).
8. R. B. Griffiths and P. A. Pearce, Mathernatical properties of position-space renormalization-group transformations, *J. Stat. Phys.* **20**:499–545 (1979).
9. R. B. Israel, Banach algebras and Kadanoff transformations, *Random Fields (Esztergom, 1979)*, Vol. II, J. Fritz, J. L. Lebowitz, and D. Szász, eds. (North-Holland, Amsterdam, 1981), pp. 593–608.
10. H. Künsch, Almost sure entropy and the variational principle for random fields with unbounded state space, *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **58**:69–85 (1981).
11. J. L. Lebowitz and E. Presutti, Statistical Mechanics of systems of unbounded spins, *Comman. Math. Phys.* **50**:195–218 (1976).
12. J. Lörinczi and C. Maes. Weakly Gibbsian measures for lattice spins systems, *J. Stat. Phys.* **89**:561–579 (1997).
13. J. Lörinczi and K. Vande Velde, A note on the projection of Gibbs measures, *J. Stat. Phys.* **77**:881–887 (1994).
14. C. Maes and K. Vande Velde. Relative energies for non-Gibbsian states, *Commun. Math. Phys.* **189**:277–286 (1997).
15. F. Martinelli and E. Olivieri, Some remarks on pathologies of renormalization-group transformations, *J. Stat. Phys.* **72**:1169–1177 (1993).
16. F. Martinelli and E. Olivieri, Instability of renormalization-group pathologies under decimation, *J. Stat. Phys.* **79**:25–42 (1995).
17. R. H. Schonmann, Projections of Gibbs measures may be non-Gibbsian, *Comm. Math. Phys.* **124**:1–7 (1989).
18. A. C. D. van Enter, Ill-defined block-spin transformations at arbitrarily high temperatures, *J. Stat. Phys.* **83**:761–765 (1996).
19. A. C. D. van Enter, R. Fernández, and R. Kotecký, Pathological behavior of renormalization group maps at high fields and above the transition temperature, *J. Stat. Phys.* **79**:969–992 (1995).
20. A. C. D. van Enter, R. Fernández, and A. D. Sokal, Renormalization transformations in the vicinity of first-order phase transitions: What can and cannot go wrong, *Phys. Rev. Lett.* **66**:3253–3256 (1991).
21. A. C. D. van Enter, R. Fernández, and A. D. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, *J. Stat. Phys.* **72**:879–1167 (1993).
22. A. C. D. van Enter and J. Lörinczi, Robustness of the non-Gibbsian property: some examples. University of Groningen preprint, 1995, *J. Phys. A, Math. and Gen.* **29**:2465–2473 (1996).